ON FLOATING BODIES.

BOOK II.

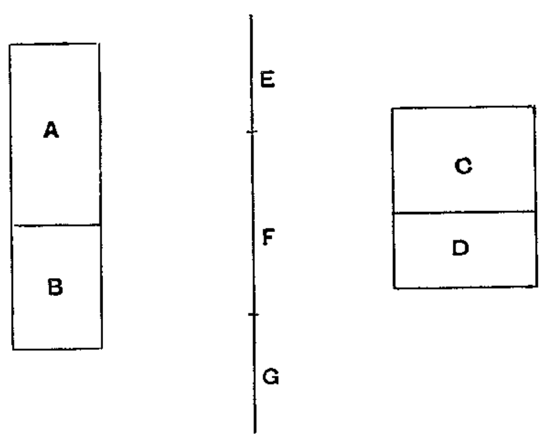
Proposition 1.

If a solid lighter than a fluid be at rest in it, the weight of the solid will be to that of the same volume of the fluid as the immersed portion of the solid is to the whole.

Let $(A+B)$ be the solid, B the portion immersed in the fluid.

Let $(C+D)$ be an equal volume of the fluid, C being equal in volume to A and B to D.

Further suppose the line E to represent the weight of the solid $(A+B)$, $(F+G)$ to represent the weight of $(C+D)$, and G that of D.



Then

weight of $(A+B)$: weight of $(C+D) = E : (F+G) \dots (1)$.

And the weight of $(A+B)$ is equal to the weight of a volume B of the fluid [I. 5], i.e. to the weight of D.

That is to say, $E=G$.

Hence, by (1),

$$\text{weight of } (A+B) : \text{weight of } (C+D) = G : F+G$$
$$= D : C+D$$
$$= B : A+B.$$

Proposition 2.

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ (where p is the principal parameter of the generating parabola), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined to the vertical at any angle, but so that the base of the segment does not touch the surface of the fluid, the segment of the paraboloid will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN, and through AN draw a plane perpendicular to the surface of the fluid. Let the plane intersect the paraboloid in the parabola BAB', the base of the segment of the paraboloid in BB', and the plane of the surface of the fluid in the chord QQ' of the parabola.

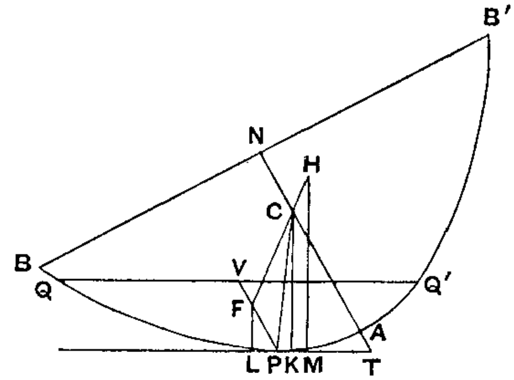
Then, since the axis AN is placed in a position not perpendicular to QQ', BB' will not be parallel to QQ'.

Draw the tangent PT to the parabola which is parallel to QQ', and let P be the point of contact*.

[From P draw PV parallel to AN meeting QQ' in V. Then PV will be a diameter of the parabola, and also the axis of the portion of the paraboloid immersed in the fluid.

* The rest of the proof is wanting in the version of Tartaglia, but is given in brackets as supplied by Commandinus.

Let C be the centre of gravity of the paraboloid BAB', and F that of the portion immersed in the fluid. Join FC and produce it to H so that H is the centre of gravity of the remaining portion of the paraboloid above the surface.



Then, since $AN = \frac{3}{2}AC$*,

and $AN \not> \frac{3}{4}p$,

it follows that $AC \not> \frac{p}{2}$.

Therefore, if CP be joined, the angle CPT is acute†. Hence, if CK be drawn perpendicular to PT, K will fall between P and T. And, if FL, HM be drawn parallel to CK to meet PT, they will each be perpendicular to the surface of the fluid.

Now the force acting on the immersed portion of the segment of the paraboloid will act upwards along LF, while the weight of the portion outside the fluid will act downwards along HM.

Therefore there will not be equilibrium, but the segment

* As the determination of the centre of gravity of a segment of a paraboloid which is here assumed does not appear in any extant work of Archimedes, or in any known work by any other Greek mathematician, it appears probable that it was investigated by Archimedes himself in some treatise now lost.

† The truth of this statement is easily proved from the property of the sub-normal. For, if the normal at P meet the axis in G, AG is greater than $\frac{p}{2}$ except in the case where the normal is the normal at the vertex A itself. But the latter case is excluded here because, by hypothesis, AN is not placed vertically. Hence, P being a different point from A, AG is always greater than AC; and, since the angle TPG is right, the angle TPC must be acute.

will turn so that B will rise and B' will fall, until AN takes the vertical position.]

[For purposes of comparison the trigonometrical equivalent of this and other propositions will be appended.

Suppose that the angle NTP, at which in the above figure the axis AN is inclined to the surface of the fluid, is denoted by θ.

Then the coordinates of P referred to AN and the tangent at A as axes are

$$\frac{p}{4}\cot^2\theta, \quad \frac{p}{2}\cot\theta,$$

where p is the principal parameter.

Suppose that $AN = h$, $PV = k$.

If now x' be the distance from T of the orthogonal projection of F on TP, and x the corresponding distance for the point C, we have

$$x' = \frac{p}{2}\cot^2\theta \,.\, \cos\theta + \frac{p}{2}\cot\theta \,.\, \sin\theta + \frac{2}{3}k\cos\theta,$$

$$x = \frac{p}{4}\cot^2\theta \,.\, \cos\theta + \frac{2}{3}h\cos\theta,$$

whence $$x' - x = \cos\theta\left\{\frac{p}{4}(\cot^2\theta + 2) - \frac{2}{3}(h-k)\right\}.$$

In order that the segment of the paraboloid may turn in the direction of increasing the angle PTN, x' must be greater than x, or the expression just found must be positive.

This will always be the case, whatever be the value of θ, if

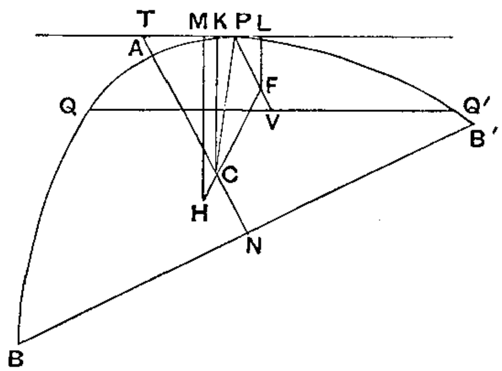
$$\frac{p}{2} \nless \frac{2h}{3},$$

or $$h \ngtr \tfrac{3}{4}p.]$$

Proposition 3.

If a right segment of a paraboloid of revolution whose axis is not greater than $\frac{3}{4}p$ *(where p is the parameter), and whose specific gravity is less than that of a fluid, be placed in the fluid with its axis inclined at any angle to the vertical, but so that its* *base is entirely submerged, the solid will not remain in that position but will return to the position in which the axis is vertical.*

Let the axis of the paraboloid be AN, and through AN draw a plane perpendicular to the surface of the fluid intersecting the paraboloid in the parabola BAB', the base of the segment in BNB', and the plane of the surface of the fluid in the chord QQ' of the parabola.



Then, since AN, as placed, is not perpendicular to the surface of the fluid, QQ' and BB' will not be parallel.

Draw PT parallel to QQ' and touching the parabola at P. Let PT meet NA produced in T. Draw the diameter PV bisecting QQ' in V. PV is then the axis of the portion of the paraboloid above the surface of the fluid.

Let C be the centre of gravity of the whole segment of the paraboloid, F that of the portion above the surface. Join FC and produce it to H so that H is the centre of gravity of the immersed portion.

Then, since $AC \not> \frac{p}{2}$, the angle CPT is an acute angle, as in the last proposition.

Hence, if CK be drawn perpendicular to PT, K will fall between P and T. Also, if HM, FL be drawn parallel to CK, they will be perpendicular to the surface of the fluid.

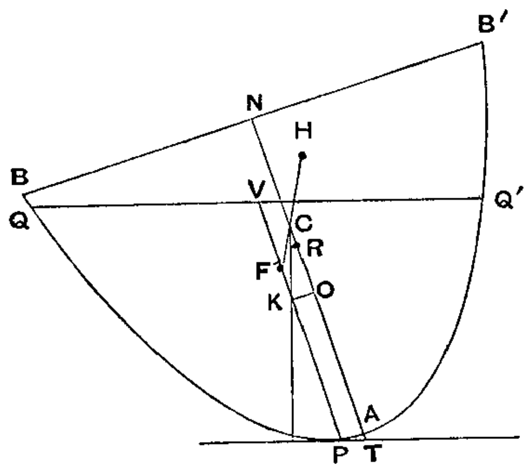
And the force acting on the submerged portion will act upwards along HM, while the weight of the rest will act downwards along LF produced.

Thus the paraboloid will turn until it takes the position in which AN is vertical.

Proposition 4.

Given a right segment of a paraboloid of revolution whose axis AN is greater than $\frac{3}{4}p$ (where p is the parameter), and whose specific gravity is less than that of a fluid but bears to it a ratio not less than $(AN - \frac{3}{4}p)^2 : AN^2$, if the segment of the paraboloid be placed in the fluid with its axis at any inclination to the vertical, but so that its base does not touch the surface of the fluid, it will not remain in that position but will return to the position in which its axis is vertical.

Let the axis of the segment of the paraboloid be AN, and let a plane be drawn through AN perpendicular to the surface of the fluid and intersecting the segment in the parabola BAB', the base of the segment in BB', and the surface of the fluid in the chord QQ' of the parabola.



Then AN, as placed, will not be perpendicular to QQ'.

Draw PT parallel to QQ' and touching the parabola at P. Draw the diameter PV bisecting QQ' in V. Thus PV will be the axis of the submerged portion of the solid.

Let C be the centre of gravity of the whole solid, F that of the immersed portion. Join FC and produce it to H so that H is the centre of gravity of the remaining portion.

Now, since $\quad AN = \frac{3}{2}AC$,

and $\quad AN > \frac{3}{4}p$,

it follows that $\quad AC > \frac{p}{2}$.

Measure CO along CA equal to $\frac{p}{2}$, and OR along OC equal to $\frac{1}{2}AO$.

Then, since $$AN = \tfrac{3}{2}AC,$$
and $$AR = \tfrac{3}{2}AO,$$
we have, by subtraction,
$$NR = \tfrac{3}{2}OC.$$

That is, $$AN - AR = \tfrac{3}{2}OC$$
$$= \tfrac{3}{4}p,$$
or $$AR = (AN - \tfrac{3}{4}p).$$
Thus $$(AN - \tfrac{3}{4}p)^2 : AN^2 = AR^2 : AN^2,$$
and therefore the ratio of the specific gravity of the solid to that of the fluid is, by the enunciation, not less than the ratio $AR^2 : AN^2$.

But, by Prop. 1, the former ratio is equal to the ratio of the immersed portion to the whole solid, i.e. to the ratio $PV^2 : AN^2$ [*On Conoids and Spheroids*, Prop. 24].

Hence $$PV^2 : AN^2 \nless AR^2 : AN^2,$$
or $$PV \nless AR.$$

It follows that
$$PF \left(= \tfrac{2}{3}PV\right) \nless \tfrac{2}{3}AR$$
$$\nless AO.$$
If, therefore, OK be drawn from O perpendicular to OA, it will meet PF between P and F.

Also, if CK be joined, the triangle KCO is equal and similar to the triangle formed by the normal, the subnormal and the ordinate at P (since $CO = \frac{1}{2}p$ or the subnormal, and KO is equal to the ordinate).

Therefore CK is parallel to the normal at P, and therefore perpendicular to the tangent at P and to the surface of the fluid.

Hence, if parallels to CK be drawn through F, H, they will be perpendicular to the surface of the fluid, and the force acting on the submerged portion of the solid will act upwards along the former, while the weight of the other portion will act downwards along the latter.