BOOK THREE

DEFINITIONS

1. *Equal circles* are those the diameters of which are equal, or the radii of which are equal.

2. A straight line is said to *touch a circle* which, meeting the circle and being produced, does not cut the circle.

3. *Circles* are said to *touch one another* which, meeting one another, do not cut one another.

4. In a circle straight lines are said *to be equally distant from the centre* when the perpendiculars drawn to them from the centre are equal.

5. And that straight line is said to be *at a greater distance* on which the greater perpendicular falls.

6. A *segment of a circle* is the figure contained by a straight line and a circumference of a circle.

7. An *angle of a segment* is that contained by a straight line and a circumference of a circle.

8. An *angle in a segment* is the angle which, when a point is taken on the circumference of the segment and straight lines are joined from it to the extremities of the straight line which is the *base of the segment*, is contained by the straight lines so joined.

9. And, when the straight lines containing the angle cut off a circumference, the angle is said to *stand upon* that circumference.

10. A *sector of a circle* is the figure which, when an angle is constructed at the centre of the circle, is contained by the straight lines containing the angle and the circumference cut off by them.

11. *Similar segments of circles* are those which admit equal angles, or in which the angles are equal to one another.

BOOK III. PROPOSITIONS

Proposition 1

To find the centre of a given circle.

Let ABC be the given circle;

thus it is required to find the centre of the circle ABC.

Let a straight line AB be drawn through it at random, and let it be bisected at the point D;
from D let DC be drawn at right angles to AB and let it be drawn through to E; let CE be bisected at F;

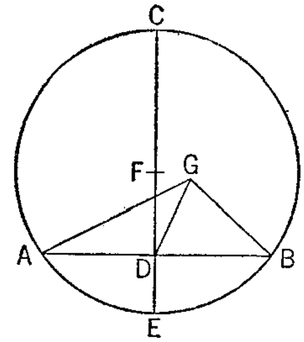
I say that F is the centre of the circle ABC.

For suppose it is not, but, if possible, let G be the centre, and let GA, GD, GB be joined.

Then, since AD is equal to DB, and DG is common,

the two sides AD, DG are equal to the two sides BD, DG respectively; and the base GA is equal to the base GB, for they are radii;

therefore the angle ADG is equal to the angle GDB. [I. 8]



But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore the angle GDB is right.

But the angle FDB is also right;

therefore the angle FDB is equal to the angle GDB, the greater to the less: which is impossible.

Therefore G is not the centre of the circle ABC.

Similarly we can prove that neither is any other point except F.

Therefore the point F is the centre of the circle ABC.

Porism. From this it is manifest that, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line. Q. E. F.

Proposition 2

If on the circumference of a circle two points be taken at random, the straight line joining the points will fall within the circle.

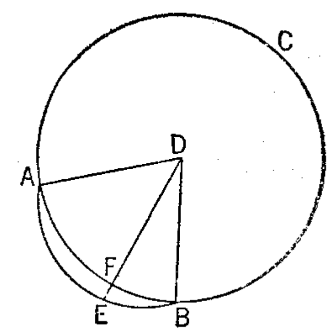
Let ABC be a circle, and let two points A, B be taken at random on its circumference;

I say that the straight line joined from A to B will fall within the circle.

For suppose it does not, but, if possible, let it fall outside, as AEB; let the centre of the circle ABC be taken [III. 1], and let it be D; let DA, DB be joined, and let DFE be drawn through.

Then, since DA is equal to DB,

the angle DAE is also equal to the angle DBE. [I. 5]



And, since one side AEB of the triangle DAE is produced,

the angle DEB is greater than the angle DAE. [I. 16]

But the angle DAE is equal to the angle DBE;

therefore the angle DEB is greater than the angle DBE.

And the greater angle is subtended by the greater side; [I. 19]

therefore DB is greater than DE.

But DB is equal to DF;

therefore DF is greater than DE,

the less than the greater: which is impossible.

Therefore the straight line joined from A to B will not fall outside the circle.

Similarly we can prove that neither will it fall on the circumference itself;

therefore it will fall within.

Therefore etc. Q. E. D.

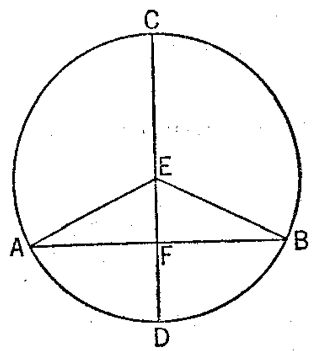
Proposition 3

If in a circle a straight line through the centre bisect a straight line not through the centre, it also cuts it at right angles; and if it cut it at right angles, it also bisects it.

Let ABC be a circle, and in it let a straight line CD through the centre bisect a straight line AB not through the centre at the point F;

I say that it also cuts it at right angles.

For let the centre of the circle ABC be taken, and let it be E; let EA, EB be joined.



Then, since AF is equal to FB, and FE is common, two sides are equal to two sides;

and the base EA is equal to the base EB;

therefore the angle AFE is equal to the angle BFE. [I. 8]

But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; [I. Def. 10]

therefore each of the angles AFE, BFE is right.

Therefore CD, which is through the centre, and bisects AB which is not through the centre, also cuts it at right angles.

Again, let CD cut AB at right angles;

I say that it also bisects it, that is, that AF is equal to FB.

For, with the same construction,

since EA is equal to EB,

the angle EAF is also equal to the angle EBF. [I. 5]

But the right angle AFE is equal to the right angle BFE, therefore EAF, EBF are two triangles having two angles equal to two angles and one side equal to one side, namely EF, which is common to them, and subtends one of the equal angles;

therefore they will also have the remaining sides equal to the remaining sides; [I. 26]

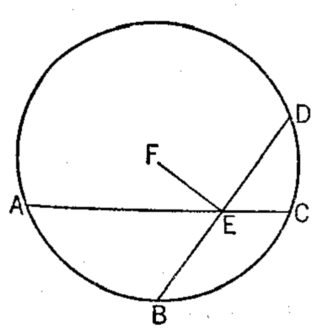
therefore AF is equal to FB.

Therefore etc. Q. E. D.

Proposition 4

If in a circle two straight lines cut one another which are not through the centre, they do not bisect one another.

Let $ABCD$ be a circle, and in it let the two straight lines AC, BD, which are not through the centre, cut one another at E;



I say that they do not bisect one another.

For, if possible, let them bisect one another, so that AE is equal to EC, and BE to ED;

let the centre of the circle $ABCD$ be taken [III. 1], and let it be F; let FE be joined.

Then, since a straight line FE through the centre bisects a straight line AC not through the centre,

it also cuts it at right angles; [III. 3]

therefore the angle FEA is right.

Again, since a straight line FE bisects a straight line BD,
it also cuts it at right angles; [III. 3]
therefore the angle FEB is right.

But the angle FEA was also proved right;
therefore the angle FEA is equal to the angle FEB, the less to the greater: which is impossible.

Therefore AC, BD do not bisect one another.

Therefore etc. Q. E. D.

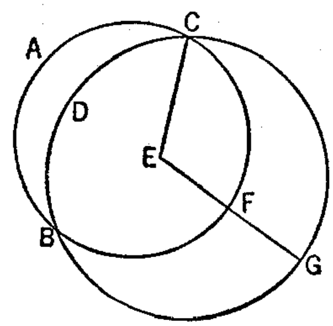
Proposition 5

If two circles cut one another, they will not have the same centre.

For let the circles ABC, CDG cut one another at the points B, C;

I say that they will not have the same centre.

For, if possible, let it be E; let EC be joined, and let EFG be drawn through at random.



Then, since the point E is the centre of the circle ABC,
EC is equal to EF. [I. Def. 15]

Again, since the point E is the centre of the circle CDG,
EC is equal to EG.

But EC was proved equal to EF also;
therefore EF is also equal to EG, the less to the greater: which is impossible.

Therefore the point E is not the centre of the circles ABC, CDG.

Therefore etc. Q. E. D.

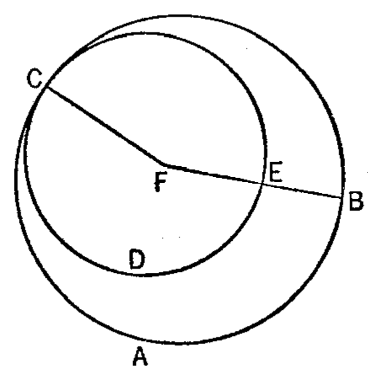
Proposition 6

If two circles touch one another, they will not have the same centre.

For let the two circles ABC, CDE touch one another at the point C;

I say that they will not have the same centre.

For, if possible, let it be F; let FC be joined, and let FEB be drawn through at random.



Then, since the point F is the centre of the circle ABC,
FC is equal to FB.

Again, since the point F is the centre of the circle CDE,
FC is equal to FE.

But FC was proved equal to FB;
therefore FE is also equal to FB, the less to the greater: which is impossible.

Therefore F is not the centre of the circles ABC, CDE.

Therefore etc. Q. E. D.

Proposition 7

If on the diameter of a circle a point be taken which is not the centre of the circle, and from the point straight lines fall upon the circle, that will be greatest on which the centre is, the remainder of the same diameter will be least, and of the rest the *nearer to the straight line through the centre is always greater than the more remote, and only two equal straight lines will fall from the point on the circle, one on each side of the least straight line.*

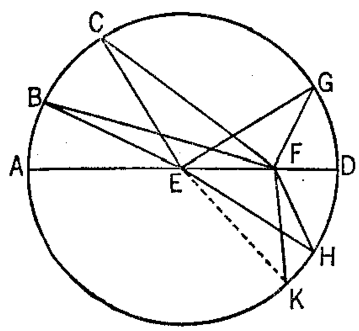
Let $ABCD$ be a circle, and let AD be a diameter of it; on AD let a point F be taken which is not the centre of the circle, let E be the centre of the circle,

and from F let straight lines FB, FC, FG fall upon the circle $ABCD$;

I say that FA is greatest, FD is least, and of the rest FB is greater than FC, and FC than FG.

For let BE, CE, GE be joined.

Then, since in any triangle two sides are greater than the remaining one, [I. 20]



EB, EF are greater than BF.

But AE is equal to BE;

therefore AF is greater than BF.

Again, since BE is equal to CE, and FE is common, the two sides BE, EF are equal to the two sides CE, EF.

But the angle BEF is also greater than the angle CEF;

therefore the base BF is greater than the base CF. [I. 24]

For the same reason

CF is also greater than FG.

Again, since GF, FE are greater than EG,

and EG is equal to ED,

GF, FE are greater than ED.

Let EF be subtracted from each;

therefore the remainder GF is greater than the remainder FD.

Therefore FA is greatest, FD is least, and FB is greater than FC, and FC than FG.

I say also that from the point F only two equal straight lines will fall on the circle $ABCD$, one on each side of the least FD.

For on the straight line EF, and at the point E on it, let the angle FEH be constructed equal to the angle GEF [I. 23], and let FH be joined.

Then, since GE is equal to EH,

and EF is common,

the two sides GE, EF are equal to the two sides HE, EF;

and the angle GEF is equal to the angle HEF;

therefore the base FG is equal to the base FH. [I. 4]

I say again that another straight line equal to FG will not fall on the circle from the point F.

For, if possible, let FK so fall.

Then, since FK is equal to FG, and FH to FG,

FK is also equal to FH,

the nearer to the straight line through the centre being thus equal to the more remote: which is impossible.

Therefore another straight line equal to GF will not fall from the point F upon the circle;

therefore only one straight line will so fall.

Therefore etc. Q. E. D.

Proposition 8

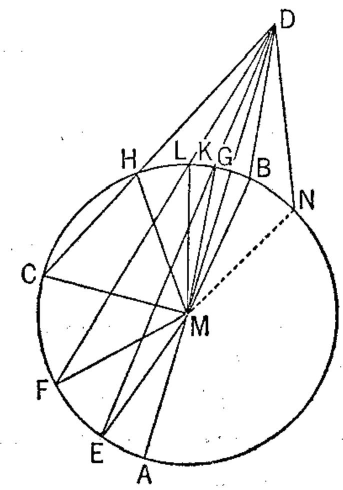
If a point be taken outside a circle and from the point straight lines be drawn through to the circle, one of which is through the centre and the others are drawn at random, then, of the straight lines which fall on the concave circumference, that through the centre is greatest, while of the rest the nearer to that through the centre is always greater than the more remote, but, of the straight lines falling on the convex circumference, that between the point and the diameter is least, while of the rest the nearer to the least is always less than the more remote, and only two equal straight lines will fall on the circle from the point, one on each side of the least.

Let ABC be a circle, and let a point D be taken outside ABC; let there be drawn through from it straight lines DA, DE, DF, DC, and let DA be through the centre;

I say that, of the straight lines falling on the concave circumference $AEFC$, the straight line DA through the centre is greatest,

while DE is greater than DF and DF than DC;

but, of the straight lines falling on the convex circumference $HLKG$, the straight line DG between the point and the diameter AG is least; and the nearer to the least DG is always less than the more remote, namely DK than DL, and DL than DH.



For let the centre of the circle ABC be taken [III. 1], and let it be M; let ME, MF, MC, MK, ML, MH be joined.

Then, since AM is equal to EM, let MD be added to each;

therefore AD is equal to EM, MD.

But EM, MD are greater than ED;

therefore AD is also greater than ED.

Again, since ME is equal to MF,

and MD is common,

therefore EM, MD are equal to FM, MD;

and the angle EMD is greater than the angle FMD;

therefore the base ED is greater than the base FD. [I. 24]

Similarly we can prove that FD is greater than CD; therefore DA is greatest, while DE is greater than DF, and DF than DC.

Next, since MK, KD are greater than MD, [I. 20]

and MG is equal to MK,

therefore the remainder KD is greater than the remainder GD,

so that GD is less than KD.

And, since on MD, one of the sides of the triangle MLD, two straight lines MK, KD were constructed meeting within the triangle,

therefore MK, KD are less than ML, LD; [I. 21]

and MK is equal to ML;

therefore the remainder DK is less than the remainder DL.

Similarly we can prove that DL is also less than DH;

therefore DG is least, while DK is less than DL, and DL than DH.

I say also that only two equal straight lines will fall from the point D on the circle, one on each side of the least DG.

On the straight line MD, and at the point M on it, let the angle DMB be constructed equal to the angle KMD, and let DB be joined.

Then, since MK is equal to MB,

and MD is common,

the two sides KM, MD are equal to the two sides BM, MD respectively;

and the angle KMD is equal to the angle BMD;

therefore the base DK is equal to the base DB. [I. 4]

I say that no other straight line equal to the straight line DK will fall on the circle from the point D.

For, if possible, let a straight line so fall, and let it be DN.

Then, since DK is equal to DN,

while DK is equal to DB,

DB is also equal to DN,

that is, the nearer to the least DG equal to the more remote: which was proved impossible.

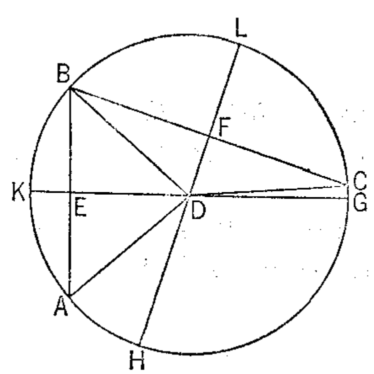
Therefore no more than two equal straight lines will fall on the circle ABC from the point D, one on each side of DG the least.

Therefore etc. Q. E. D.

PROPOSITION 9

If a point be taken within a circle, and more than two equal straight lines fall from the point on the circle, the point taken is the centre of the circle.

Let ABC be a circle and D a point within it, and from D let more than two equal straight lines, namely DA, DB, DC, fall on the circle ABC;



I say that the point D is the centre of the circle ABC.

For let AB, BC be joined and bisected at the points E, F, and let ED, FD be joined and drawn through to the points G, K, H, L.

Then, since AE is equal to EB, and ED is common,

the two sides AE, ED are equal to the two sides BE, ED;

and the base DA is equal to the base DB;

therefore the angle AED is equal to the angle BED. [I. 8]

Therefore each of the angles AED, BED is right; [I. Def. 10]

therefore GK cuts AB into two equal parts and at right angles.

And since, if in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line, [III. 1, Por.]

the centre of the circle is on GK.

For the same reason

the centre of the circle ABC is also on HL.

And the straight lines GK, HL have no other point common but the point D;

therefore the point D is the centre of the circle ABC.

Therefore etc. Q. E. D.