BOOK TWELVE

PROPOSITIONS

Proposition 1

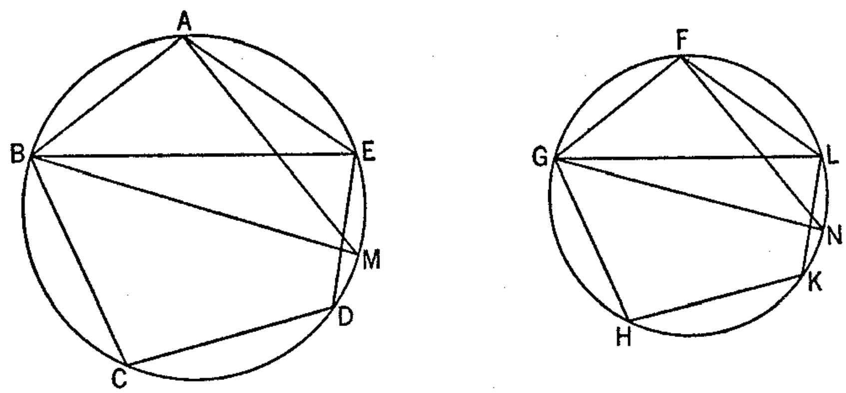
Similar polygons inscribed in circles are to one another as the squares on the diameters.

Let *ABC*, *FGH* be circles,
let *ABCDE*, *FGHKL* be similar polygons inscribed in them, and let *BM*, *GN* be diameters of the circles;

I say that, as the square on *BM* is to the square on *GN*, so is the polygon *ABCDE* to the polygon *FGHKL*.

For let *BE*, *AM*, *GL*, *FN* be joined.

Now, since the polygon *ABCDE* is similar to the polygon *FGHKL*,



the angle *BAE* is equal to the angle *GFL*,
and, as *BA* is to *AE*, so is *GF* to *FL*. [VI. Def. 1]

Thus *BAE*, *GFL* are two triangles which have one angle equal to one angle, namely the angle *BAE* to the angle *GFL*, and the sides about the equal angles proportional;
therefore the triangle *ABE* is equiangular with the triangle *FGL*. [VI. 6]

Therefore the angle *AEB* is equal to the angle *FLG*.

But the angle *AEB* is equal to the angle *AMB*,
for they stand on the same circumference; [III. 27]
and the angle *FLG* to the angle *FNG*;
therefore the angle *AMB* is also equal to the angle *FNG*.

But the right angle *BAM* is also equal to the right angle *GFN*; [III. 31]
therefore the remaining angle is equal to the remaining angle. [I. 32]

Therefore the triangle *ABM* is equiangular with the triangle *FGN*.

Therefore, proportionally, as *BM* is to *GN*, so is *BA* to *GF*. [VI. 4]

But the ratio of the square on BM to the square on GN is duplicate of the ratio of BM to GN,
and the ratio of the polygon $ABCDE$ to the polygon $FGHKL$ is duplicate of the ratio of BA to GF; [VI. 20]
therefore also, as the square on BM is to the square on GN, so is the polygon $ABCDE$ to the polygon $FGHKL$.

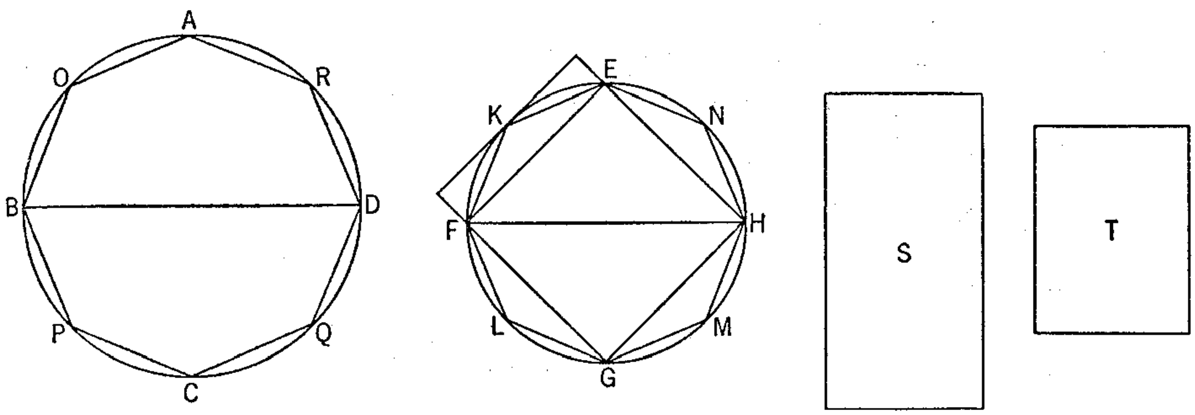
Therefore etc. Q. E. D.

PROPOSITION 2

Circles are to one another as the squares on the diameters.

Let $ABCD$, $EFGH$ be circles, and BD, FH their diameters;

I say that, as the circle $ABCD$ is to the circle $EFGH$, so is the square on BD to the square on FH.



For, if the square on BD is not to the square on FH as the circle $ABCD$ is to the circle $EFGH$,
then, as the square on BD is to the square on FH, so will the circle $ABCD$ be either to some less area than the circle $EFGH$, or to a greater.

First, let it be in that ratio to a less area S.

Let the square $EFGH$ be inscribed in the circle $EFGH$; then the inscribed square is greater than the half of the circle $EFGH$, inasmuch as, if through the points E, F, G, H we draw tangents to the circle, the square $EFGH$ is half the square circumscribed about the circle, and the circle is less than the circumscribed square;
hence the inscribed square $EFGH$ is greater than the half of the circle $EFGH$.

Let the circumferences EF, FG, GH, HE be bisected at the points K, L, M, N,

and let EK, KF, FL, LG, GM, MH, HN, NE be joined;
therefore each of the triangles EKF, FLG, GMH, HNE is also greater than the half of the segment of the circle about it, inasmuch as, if through the points K, L, M, N we draw tangents to the circle and complete the parallelograms on the straight lines EF, FG, GH, HE, each of the triangles EKF, FLG, GMH, HNE will be half of the parallelogram about it,

while the segment about it is less than the parallelogram;
hence each of the triangles EKF, FLG, GMH, HNE is greater than the half of the segment of the circle about it.

Thus, by bisecting the remaining circumferences and joining straight lines, and by doing this continually, we shall leave some segments of the circle which will be less than the excess by which the circle *EFGH* exceeds the area *S*.

For it was proved in the first theorem of the tenth book that, if two unequal magnitudes be set out, and if from the greater there be subtracted a magnitude greater than the half, and from that which is left a greater than the half, and if this be done continually, there will be left some magnitude which will be less than the lesser magnitude set out.

Let segments be left such as described, and let the segments of the circle *EFGH* on *EK*, *KF*, *FL*, *LG*, *GM*, *MH*, *HN*, *NE* be less than the excess by which the circle *EFGH* exceeds the area *S*.

Therefore the remainder, the polygon *EKFLGMHN*, is greater than the area *S*.

Let there be inscribed, also, in the circle *ABCD* the polygon *AOBPCQDR* similar to the polygon *EKFLGMHN*;
therefore, as the square on *BD* is to the square on *FH*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*. [XII. 1]

But, as the square on *BD* is to the square on *FH*, so also is the circle *ABCD* to the area *S*;
therefore also, as the circle *ABCD* is to the area *S*, so is the polygon *AOBPCQDR* to the polygon *EKFLGMHN*; [V. 11]
therefore, alternately, as the circle *ABCD* is to the polygon inscribed in it, so is the area *S* to the polygon *EKFLGMHN*. [V. 16]

But the circle *ABCD* is greater than the polygon inscribed in it;
therefore the area *S* is also greater than the polygon *EKFLGMHN*.

But it is also less:
which is impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area less than the circle *EFGH*.

Similarly we can prove that neither is the circle *EFGH* to any area less than the circle *ABCD* as the square on *FH* is to the square on *BD*.

I say next that neither is the circle *ABCD* to any area greater than the circle *EFGH* as the square on *BD* is to the square on *FH*.

For, if possible, let it be in that ratio to a greater area *S*.

Therefore, inversely, as the square on *FH* is to the square on *DB*, so is the area *S* to the circle *ABCD*.

But, as the area *S* is to the circle *ABCD*, so is the circle *EFGH* to some area less than the circle *ABCD*;
therefore also, as the square on *FH* is to the square on *BD*, so is the circle *EFGH* to some area less than the circle *ABCD*: [V. 11]
which was proved impossible.

Therefore, as the square on *BD* is to the square on *FH*, so is not the circle *ABCD* to any area greater than the circle *EFGH*.

And it was proved that neither is it in that ratio to any area less than the circle *EFGH*;
therefore, as the square on *BD* is to the square on *FH*, so is the circle *ABCD* to the circle *EFGH*.

Therefore etc. Q. E. D.

Lemma

I say that, the area S being greater than the circle $EFGH$, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$.

For let it be contrived that, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T.

I say that the area T is less than the circle $ABCD$.

For since, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to the area T,
therefore, alternately, as the area S is to the circle $EFGH$, so is the circle $ABCD$ to the area T. [v. 16]

But the area S is greater than the circle $EFGH$;

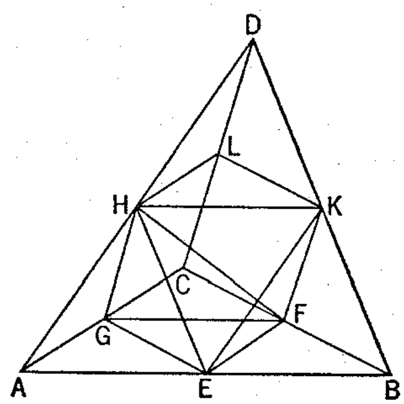
therefore the circle $ABCD$ is also greater than the area T.

Hence, as the area S is to the circle $ABCD$, so is the circle $EFGH$ to some area less than the circle $ABCD$. Q. E. D.

Proposition 3

Any pyramid which has a triangular base is divided into two pyramids equal and similar to one another, similar to the whole and having triangular bases, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

Let there be a pyramid of which the triangle ABC is the base and the point D the vertex;



I say that the pyramid $ABCD$ is divided into two pyramids equal to one another, having triangular bases and similar to the whole pyramid, and into two equal prisms; and the two prisms are greater than the half of the whole pyramid.

For let AB, BC, CA, AD, DB, DC be bisected at the points E, F, G, H, K, L, and let HE, EG, GH, HK, KL, LH, KF, FG be joined.

Since AE is equal to EB, and AH to DH,

therefore EH is parallel to DB. [vi. 2]

For the same reason

HK is also parallel to AB.

Therefore $HEBK$ is a parallelogram;

therefore HK is equal to EB. [i. 34]

But EB is equal to EA;

therefore AE is also equal to HK.

But AH is also equal to HD;
therefore the two sides EA, AH are equal to the two sides KH, HD respectively,

and the angle EAH is equal to the angle KHD;

therefore the base EH is equal to the base KD. [i. 4]

Therefore the triangle AEH is equal and similar to the triangle HKD.

For the same reason

the triangle AHG is also equal and similar to the triangle HLD.

Now, since two straight lines EH, HG meeting one another are parallel to two straight lines KD, DL meeting one another, and are not in the same plane, they will contain equal angles. [xi. 10]

Therefore the angle EHG is equal to the angle KDL.

And, since the two straight lines EH, HG are equal to the two KD, DL respectively,

and the angle EHG is equal to the angle KDL,

therefore the base EG is equal to the base KL; [I. 4]

therefore the triangle EHG is equal and similar to the triangle KDL.

For the same reason

the triangle AEG is also equal and similar to the triangle HKL.

Therefore the pyramid of which the triangle AEG is the base and the point H the vertex is equal and similar to the pyramid of which the triangle HKL is the base and the point D the vertex. [XI. Def. 10]

And, since HK has been drawn parallel to AB, one of the sides of the triangle ADB,

the triangle ADB is equiangular to the triangle DHK, [I. 29]

and they have their sides proportional;

therefore the triangle ADB is similar to the triangle DHK. [VI. Def. 1]

For the same reason

the triangle DBC is also similar to the triangle DKL, and the triangle ADC to the triangle DLH.

Now, since the two straight lines BA, AC meeting one another are parallel to the two straight lines KH, HL meeting one another, not in the same plane, they will contain equal angles. [XI. 10]

Therefore the angle BAC is equal to the angle KHL.

And, as BA is to AC, so is KH to HL;

therefore the triangle ABC is similar to the triangle HKL.

Therefore also the pyramid of which the triangle ABC is the base and the point D the vertex is similar to the pyramid of which the triangle HKL is the base and the point D the vertex.

But the pyramid of which the triangle HKL is the base and the point D the vertex was proved similar to the pyramid of which the triangle AEG is the base and the point H the vertex.

Therefore each of the pyramids $AEGH$, $HKLD$ is similar to the whole pyramid $ABCD$.

Next, since BF is equal to FC,

the parallelogram $EBFG$ is double of the triangle GFC.

And since, if there be two prisms of equal height, and one have a parallelogram as base, and the other a triangle, and if the parallelogram be double of the triangle, the prisms are equal, [XI. 39]

therefore the prism contained by the two triangles BKF, EHG, and the three parallelograms $EBFG$, $EBKH$, $HKFG$ is equal to the prism contained by the two triangles GFC, HKL and the three parallelograms $KFCL$, $LCGH$, $HKFG$.

And it is manifest that each of the prisms, namely that in which the parallelogram $EBFG$ is the base and the straight line HK is its opposite, and that in which the triangle GFC is the base and the triangle HKL its opposite, is greater than each of the pyramids of which the triangles AEG, HKL are the bases and the points H, D the vertices,

inasmuch as, if we join the straight lines EF, EK, the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is greater than the pyramid of which the triangle EBF is the base and the point K the vertex.

But the pyramid of which the triangle EBF is the base and the point K the vertex is equal to the pyramid of which the triangle AEG is the base and the point H the vertex;

for they are contained by equal and similar planes.

Hence also the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is greater than the pyramid of which the triangle AEG is the base and the point H the vertex.

But the prism in which the parallelogram $EBFG$ is the base and the straight line HK its opposite is equal to the prism in which the triangle GFC is the base and the triangle HKL its opposite,

and the pyramid of which the triangle AEG is the base and the point H the vertex is equal to the pyramid of which the triangle HKL is the base and the point D the vertex.

Therefore the said two prisms are greater than the said two pyramids of which the triangles AEG, HKL are the bases and the points H, D the vertices.

Therefore the whole pyramid, of which the triangle ABC is the base and the point D the vertex, has been divided into two pyramids equal to one another and into two equal prisms, and the two prisms are greater than the half of the whole pyramid. Q. E. D.

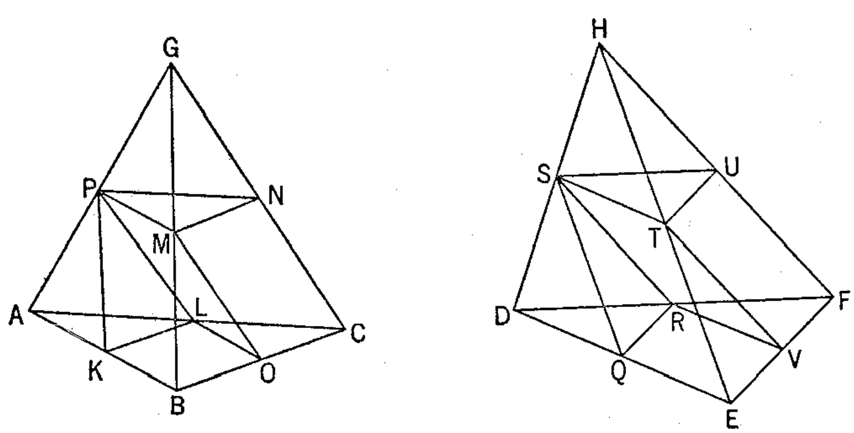
Proposition 4

If there be two pyramids of the same height which have triangular bases, and each of them be divided into two pyramids equal to one another and similar to the whole, and into two equal prisms, then, as the base of the one pyramid is to the base of the other pyramid, so will all the prisms in the one pyramid be to all the prisms, being equal in multitude, in the other pyramid.

Let there be two pyramids of the same height which have the triangular bases ABC, DEF, and vertices the points G, H,

and let each of them be divided into two pyramids equal to one another and similar to the whole and into two equal prisms; [XII. 3]

I say that, as the base ABC is to the base DEF, so are all the prisms in the pyramid $ABCG$ to all the prisms, being equal in multitude, in the pyramid $DEFH$,



For, since BO is equal to OC, and AL to LC,

therefore LO is parallel to AB,

and the triangle ABC is similar to the triangle LOC.

For the same reason

the triangle *DEF* is also similar to the triangle *RVF*.

And, since *BC* is double of *CO*, and *EF* of *FV*,

therefore, as *BC* is to *CO*, so is *EF* to *FV*.

And on *BC*, *CO* are described the similar and similarly situated rectilineal figures *ABC*, *LOC*,

and on *EF*, *FV* the similar and similarly situated figures *DEF*, *RVF*;

therefore, as the triangle *ABC* is to the triangle *LOC*, so is the triangle *DEF* to the triangle *RVF*; [vi. 22]

therefore, alternately, as the triangle *ABC* is to the triangle *DEF*, so is the triangle *LOC* to the triangle *RVF*. [v. 16]

But, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite, to the prism in which the triangle *RVF* is the base and *STU* its opposite; [Lemma following]

therefore also, as the triangle *ABC* is to the triangle *DEF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite, to the prism in which the triangle *RVF* is the base and *STU* its opposite.

But, as the said prisms are to one another, so is the prism in which the parallelogram *KBOL* is the base and the straight line *PM* its opposite, to the prism in which the parallelogram *QEVR* is the base and the straight line *ST* its opposite. [xi. 39; cf. xii. 3]

Therefore also the two prisms, that in which the parallelogram *KBOL* is the base and *PM* its opposite, and that in which the triangle *LOC* is the base and *PMN* its opposite, are to the prisms in which *QEVR* is the base and the straight line *ST* its opposite and in which the triangle *RVF* is the base and *STU* its opposite in the same ratio. [v. 12]

Therefore also, as the base *ABC* is to the base *DEF*, so are the said two prisms to the said two prisms.

And similarly, if the pyramids *PMNG*, *STUH* be divided into two prisms and two pyramids,

as the base *PMN* is to the base *STU*, so will the two prisms in the pyramid *PMNG* be to the two prisms in the pyramid *STUH*.

But, as the base *PMN* is to the base *STU*, so is the base *ABC* to the base *DEF*;

for the triangles *PMN*, *STU* are equal to the triangles *LOC*, *RVF* respectively.

Therefore also, as the base *ABC* is to the base *DEF*, so are the four prisms to the four prisms.

And similarly also, if we divide the remaining pyramids into two pyramids and into two prisms, then, as the base *ABC* is to base the *DEF*, so will all the prisms in the pyramid *ABCG* be to all the prisms, being equal in multitude, in the pyramid *DEFH*. Q. E. D.

Lemma

But that, as the triangle *LOC* is to the triangle *RVF*, so is the prism in which the triangle *LOC* is the base and *PMN* its opposite, to the prism in which the triangle *RVF* is the base and *STU* its opposite, we must prove as follows.

For in the same figure let perpendiculars be conceived drawn from *G*, *H* to

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